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# ELIMINATION OF QUANTIFIERS FOR MODULES

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ABSTRACT

Every first-order formula in the language of $R$-modules ($R$ an associative ring) is equivalent relative to the theory of $R$-modules to a boolean combination of positive primitive formulas and $\forall\exists$-sentence.

## 0. Introduction

Let $R$ be an arbitrary associative ring with 1, and let $L_R$ be the language of left $R$-modules. (For details concerning $L_R$ see, e.g., Eklof–Sabbagh [3]). In the following "module" always means "left $R$ module". Let $T_R$ be the $L_R$-theory of $R$-modules. By a positive primitive (p.p.) formula we mean an $L_R$-formula of the form $\exists \vec{y} \varphi(\vec{x}, \vec{y})$ where $\varphi(\vec{x}, \vec{y})$ is a conjunction of atomic formulas. Formulas without free variables are called sentences.

In this paper we prove the following:

THEOREM. *Every $L_R$-formula is equivalent relative to $T_R$ to a boolean combination of $\forall\exists$-sentences and positive primitive formulas.*

This Theorem generalizes in a natural way the corresponding result for abelian groups (Szmielew [5], see also Eklof–Fisher [2]). If we introduce a new $n$-place predicate symbol $R_\varphi$ for every p.p. formula with $n$ free variables and if we adjoin to $T_R$ all sentences of the form $\forall \vec{x}(R_\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x}))$, $\varphi(\vec{x})$ p.p., then the Theorem says that this enlarged theory admits elimination of quantifiers modulo certain sentences.

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## 1. Proof of the Theorem

Sabbagh [4] has shown that every $L_R$-sentence is equivalent relative to $T_R$ to a boolean combination of $\forall\exists$-sentences. Therefore it suffices to prove the following weaker statement:

(*) Every $L_R$-formula is equivalent relative to $T_R$ to a boolean combination of sentences and p.p. formulas.

REMARK. Although a slight modification of our proof would give a direct proof of the Theorem we prefer to prove (*) only, for reasons of simplicity.

TERMINOLOGY. Let $\vec{a} = \langle a_0, \cdots, a_{n-1}\rangle$ be an $n$-tuple of elements from some module $M$. The set of all p.p. formulas $\varphi(x_0, \cdots, x_{n-1})$ such that $M \models \varphi(\vec{a})$ is called the p.p. type of $\vec{a}$. An isomorphism between submodules $A$, $A'$ of $M$ is called positive primitive if $M \models \varphi(\vec{a})$ if and only if $M \models \varphi(f(\vec{a}))$ for every p.p. formula $\varphi(\vec{x})$ and every $\vec{a}$ from $A$. It is easy to see that two $n$-tuples $\vec{a}$, $\vec{a}'$ realize the same p.p. type if and only if there exists a p.p. isomorphism $f$ from the submodule generated by the $a_i$'s onto the submodule generated by the $a_i'$'s such that $f(a_i) = a_i'$. Whenever we are given $n$-tuples $\vec{a}$, $\vec{a}'$ realizing the same p.p. type we will write $A$ for $\Sigma_{i<n} Ra_i$, $A'$ for $\Sigma_{i<n} Ra_i'$ and $a'$ for $f(a)$ $(a \in A)$. This makes sense since $f$ is unique.

Finally note that all p.p. formulas $\varphi(\vec{x})$ have the following additivity property: $T_R \vdash \forall \vec{x}, \vec{y}(\varphi(\vec{x}) \& \varphi(\vec{y}) \rightarrow \varphi(\vec{x} + \vec{y}))$ and $T_R \vdash \forall \vec{x}(\varphi(\vec{x}) \rightarrow \varphi(\lambda \vec{x}))$ for all $\lambda \in R$.

In this section we will prove (*) modulo the following Lemma whose proof is postponed to the next section:

LEMMA 1. *Let $\vec{a}$, $\vec{a}'$ be $n$-tuples of elements from some module $M$ realizing the same p.p. type. Let $\varphi(\vec{x}, y)$ be a p.p. formula, and let $\psi(\vec{x}, y)$ be a conjunction of negations of p.p. formulas. Then $M \models \exists y(\varphi(\vec{a}, y)\ \&\ \psi(\vec{a}, y))$ if and only if $M \models \exists y(\varphi(\vec{a}', y)\ \&\ \psi(\vec{a}', y))$.*

LEMMA 2. *If two $n$-tuples $\vec{a}$, $\vec{a}'$ of elements from some module $M$ realize the same p.p. type then they realize the same elementary type.*

PROOF. Replacing $M$ by some elementary extension we may assume that $M$ is countably saturated. We construct a Karp-isomorphism between the structures $\langle M, \vec{a}\rangle$ and $\langle M, \vec{a}'\rangle$ (see, e.g., Barwise [1]).

Let $f$ be the unique p.p. isomorphism $A \rightarrow A'$ such that $f(a_i) = a_i'$, and let $I$ be the set of all p.p. isomorphisms $g \supseteq f$ between finitely generated submodules of $M$. By symmetry it suffices to show: if $g \in I$ and $c \in M$ then there exists $h \in I$ such that $g \subseteq h$ and $c \in \mathrm{dom}(h)$.

Put $B = \mathrm{dom}(g)$ and let $b_0, \cdots, b_{m-1}$ be a system of generators for $B$. Let $\Sigma(x_0, \cdots, x_{m-1}, y)$ be the set of all conjunctions $\chi(x_0, \cdots, x_{m-1}, y)$ of p.p. and negated p.p. formulas such that $M \models \chi(\vec{b}, c)$. By Lemma 1 $M \models \exists y \chi(g(\vec{b}), y)$. Therefore, using saturation, there exists $c' \in M$ such that $\langle g(\vec{b}), c' \rangle$ realizes $\Sigma(\vec{x}, y)$. This implies that $\langle \vec{b}, c \rangle$, $\langle g(\vec{b}), c' \rangle$ realize the same p.p. type, and hence there exists $h \in I$ such that $h \supseteq g$, $c \in \mathrm{dom}(h)$ and $h(c) = c'$.

Proof of (*). Let $\psi(\vec{x})$ be an arbitrary formula. Let $\Gamma(\vec{x})$ be the set of all formulas $\varphi(\vec{x})$ which are boolean combinations of sentences and p.p. formulas such that $T_R \vdash \varphi(\vec{x}) \rightarrow \psi(\vec{x})$. By compactness it suffices to show that $T = T_R \cup \{\psi(\vec{c})\} \cup \{\neg \varphi(\vec{c}) \mid \varphi(\vec{x}) \in \Gamma(\vec{x})\}$ is inconsistent, where $\vec{c}$ is an $n$-tuple of new constants. Assume there exists a model $\langle M, \vec{a} \rangle$ of $T$. Applying Lemma 2 and compactness we obtain a sentence $\alpha$ and a conjunction $\beta(\vec{x})$ of p.p. and negated p.p. formulas such that $M \models \alpha \mathbin{\&} \beta(\vec{a})$ and $T_R \vdash \alpha \mathbin{\&} \beta(\vec{x}) \rightarrow \psi(\vec{x})$. Therefore $\alpha \mathbin{\&} \beta(\vec{x}) \in \Gamma(\vec{x})$ hence $M \models \neg(\alpha \mathbin{\&} \beta(\vec{a}))$, contradiction.

## 2. Proof of Lemma 1

We will need two auxiliary results.

Lemma 3. *Let $M_0, \cdots, M_{k-1}$ be $R$-modules and let $\langle c_0, \cdots, c_{k-1} \rangle \in M = \bigoplus_{\kappa < k} M_\kappa$. If $M'$ is a submodule of $M$ such that the image of $M'$ under the natural projection $\pi_\kappa : M \rightarrow M_\kappa$ contains more than $k$ elements for every $\kappa < k$, then $M'$ contains an element $\langle b_0, \cdots, b_{k-1} \rangle$ such that $b_\kappa \neq c_\kappa$ for all $\kappa < k$.*

The proof is by induction on $k$. The case $k = 1$ is trivial.

$k > 1$. *Case* 1. *$M'$ infinite*

It is easy to see that there exist an infinite subset $S$ of $M'$ and a nonempty subset $I$ of $\{0, \cdots, k-1\}$ such that whenever $\vec{b} = \langle b_0, \cdots, b_{k-1} \rangle \in S$ and $\vec{b}' = \langle b'_0, \cdots, b'_{k-1} \rangle \in S$, $\vec{b} \neq \vec{b}'$, then $b_\kappa \neq b'_\kappa$ for all $\kappa \in I$ and $b_\kappa = b'_\kappa = 0$ for all $\kappa \notin I$. Applying the induction hypothesis to $N = \bigoplus_{\kappa < k, \kappa \notin I} M_\kappa$ and $N' = \pi(M')$ where $\pi : M \rightarrow N$ is the projection we obtain an element $\vec{b}' = \langle b'_0, \cdots, b'_{k-1} \rangle \in M'$ such that $b_\kappa \neq c_\kappa$ for all $\kappa \notin I$. By adding a suitable element from $S$ to $\vec{b}'$ we get an element $\vec{b} \in M'$ satisfying $b_\kappa \neq c_\kappa$ for all $\kappa < k$.

*Case* 2. *$M'$ finite*

Let $m = \mathrm{card}(M')$. Assume that for every $\vec{b} = \langle b_0, \cdots, b_{k-1} \rangle \in M'$ we have $b_\kappa = c_\kappa$ for some $\kappa$.

Put $m_\kappa = \mathrm{card}((\ker \pi_\kappa) \cap M')$, $l_\kappa = \mathrm{card}\, \pi_\kappa(M')$. Clearly $m = m_\kappa l_\kappa$ and therefore $m_\kappa = m/l_\kappa < m/k$. Since there are at most $m_\kappa$ elements in $M'$ whose $\kappa$-th coordinate is $c_\kappa$ and since $M' = \bigcup_{\kappa<k} \pi_\kappa^{-1}(c_\kappa)$ by assumption, we obtain

$$m \leqq \sum_{\kappa<k} \operatorname{card} \pi_\kappa^{-1}(c_\kappa) \leqq \sum_{\kappa<k} m_\kappa < k\frac{m}{k} = m,$$

contradiction.

LEMMA 4. *Let $\vec{a}$, $\vec{a}'$ be n-tuples of elements from some countably saturated module M, and let $\psi(\vec{x}, y)$ be an arbitrary formula. Assume that for all p.p. formulas $\varphi(\vec{x}, y)$, $M \models \exists y(\varphi(\vec{a}, y)\ \&\ \psi(\vec{a}, y))$ if and only if $M \models \exists y(\varphi(\vec{a}', y)\ \&\ \psi(\vec{a}', y))$. If $M \models \exists y \psi(\vec{a}, y)$ then there exist $d, e \in M$ such that*

i) *$\langle \vec{a}, d\rangle$, $\langle \vec{a}', e\rangle$ realize the same p.p. type,*

ii) *$M \models \psi(\vec{a}, d)$.*

PROOF. For $b \in M$ let $\Sigma_b$ be the p.p. type of $\langle \vec{a}, b\rangle$. By saturation the set $\{\Sigma_b \mid M \models \psi(\vec{a}, b)\}$ contains a maximal element $\Sigma_d$, $d \in M$. Using the hypothesis and saturation again we obtain an element $e \in M$ such that $M \models \varphi(\vec{a}', e)\ \&\ \psi(\vec{a}', e)$ for all $\varphi(\vec{x}, y) \in \Sigma_d$. Let $\varphi(\vec{x}, y)$ be a p.p. formula such that $M \models \varphi(\vec{a}', e)$. Arguing as above we see that $\Sigma_d(\vec{a}, y) \cup \{\varphi(\vec{a}, y)\} \cup \{\psi(\vec{a}, y)\}$ is consistent. Hence $\varphi(\vec{x}, y) \in \Sigma_d$, by maximality. Therefore $\langle \vec{a}, d\rangle$ and $\langle \vec{a}', e\rangle$ realize the same p.p. type.

PROOF OF LEMMA 1. We proceed by induction on the number $k$ of conjuncts in $\psi(\vec{x}, y)$. The case $k = 0$ is just the hypothesis. Let $k > 0$ and assume Lemma 1 to be true for all $k' < k$ and all $n$. By symmetry it suffices to show that

$$M \models \exists y(\varphi(\vec{a}, y)\ \&\ \psi(\vec{a}, y)) \text{ implies } M \models \exists y(\varphi(\vec{a}', y)\ \&\ \psi(\vec{a}', y)).$$

Therefore let $b_1 \in M$ such that

$$(1) \qquad M \models \varphi(\vec{a}, b_1)\ \&\ \psi(\vec{a}, b_1).$$

Furthermore we may assume that $M$ is countably saturated.

Write $\psi(\vec{x}, y)$ as $\bigwedge_{\kappa<k} \neg \psi_\kappa(\vec{x}, y)$, $\psi_\kappa(\vec{x}, y)$ p.p.

Put

$$N = \{d \in M \mid M \models \varphi(\vec{0}, d)\},$$

$$P_\kappa = M^{n+1}/\{\langle \vec{c}, d\rangle \mid M \models \psi_\kappa(\vec{c}, d)\},$$

and let $q_\kappa: M^{n+1} \to P_\kappa$ be the natural projection. Put $P = \bigoplus_{\kappa<k} P_\kappa$ and let $Q$ be the image of $N$ under the map $M \to P$ defined by

$$d \mapsto \langle q_0(\langle \vec{0}, d\rangle), \cdots, q_{k-1}(\langle \vec{0}, d\rangle)\rangle.$$

*Case* 1. The image of $Q$ under the projection $P \to P_\kappa$ contains more than $k$ elements for each $\kappa < k$.

Since $\vec{a}$, $\vec{a}'$ realize the same p.p. type there exists $b \in M$ such that $M \models \varphi(\vec{a}', b)$. Applying Lemma 3 ($M_\kappa = P_\kappa$, $M' = Q$) we find $d \in N$ such that $q_\kappa(\vec{0}, d) \neq q_\kappa(\vec{a}', b)$ for all $\kappa < k$, in other words $M \models \bigwedge_{\kappa<k} \neg \psi_\kappa(\vec{a}', b - d)$. Since $M \models \varphi(\vec{a}', b - d)$ by additivity we conclude $M \models \exists y(\varphi(\vec{a}', y) \;\&\; \psi(\vec{a}', y))$ as desired.

*Case* 2. There exist $\lambda < k$, $h \leqq k$ such that the image $J$ of $Q$ under the projection $P \to P_\lambda$ contains exactly $h$ elements.

Let $J'$ be the set consisting of those elements $p_\lambda \in J$ such that there exists $d \in A \cap N$ with $q_\lambda(\langle \vec{0}, d\rangle) = p_\lambda$. Let $h'(\leqq h)$ be the cardinality of $J'$ and choose $d_0, \cdots, d_{h'-1} \in A \cap N$ such that $\{q_\lambda(\langle \vec{0}, d_i\rangle) \mid i < h'\} = J'$.

*Case* 2.1. $h - h' = 0$, i.e. $J' = J$. We may assume that there exists an element $b \in A$ such that $M \models \varphi(\vec{a}, b)$. (If $A$ does not contain such an element we can adjoin one by applying Lemma 4 with $\varphi(\vec{x}, y)$ as $\psi(\vec{x}, y)$). The element $b_1 - b$ lies in $N$, by the additivity of $\varphi(\vec{x}, y)$. Since $J' = J$ there exists $d \in A \cap N$ such that $q_\lambda(\langle \vec{0}, d\rangle) = q_\lambda(\langle \vec{0}, b_1 - b\rangle)$. Therefore if $a_n = b + d$ then

$$M \models \psi_\lambda(\vec{0}, b_1 - a_n). \tag{2}$$

Let $\chi(x_0, \cdots, x_n, y)$ be the formula

$$\varphi(x_0, \cdots, x_{n-1}, y) \;\&\; \psi_\lambda(\vec{0}, y - x_n) \;\&\; \bigwedge_{\substack{\kappa<k \\ \kappa\neq\lambda}} \neg \psi_\kappa(\vec{x}, y).$$

By (1) and (2), $M \models \chi(\vec{a}, a_n, b_1)$ and therefore $M \models \exists y \chi(\vec{a}, a_n, y)$. Since $a_n \in A$, the $(n+1)$-tuples $\langle a_0, \cdots, a_n\rangle$, $\langle a'_0, \cdots, a'_n\rangle$ realize the same p.p. type. (Recall that $a'_n$ is the image of $a_n$ under the p.p. isomorphism $a_\nu \mapsto a'_\nu$ $(\nu < n)$). Since $\chi(\vec{x}, y)$ contains only $k - 1$ conjuncts being negations of p.p. formulas we conclude $M \models \exists y \chi(\vec{a}', a'_n, y)$, by induction hypothesis. Let $b_2 \in M$ such that $M \models \chi(\vec{a}', a'_n, b_2)$. This means

$$M \models \varphi(\vec{a}', b_2) \;\&\; \bigwedge_{\substack{\kappa<k \\ \kappa\neq\lambda}} \neg \psi_\kappa(\vec{a}', b_2) \text{ and}$$

(3) $$M \models \psi_\lambda(\vec{0}, b_2 - a'_n).$$

It remains to show that $M \models \neg\psi_\lambda(\vec{a}', b_2)$. If $M \models \psi_\lambda(\vec{a}', b_2)$ then $M \models \psi_\lambda(\vec{a}', a'_n)$ by (3) and additivity, hence $M \models \psi_\lambda(\vec{a}, a_n)$, and therefore $M \models \psi_\lambda(\vec{a}, b_1)$ by (2), contradicting (1).

*Case* 2.2. $h - h' > 0$. We reduce this case to the former by successively adjoining elements from $M$ to $A$ and $A'$. Clearly it suffices to prove the following: There exist $d, e \in N$ such that

i) $\langle \vec{a}, d\rangle$, $\langle \vec{a}', e\rangle$ realize the same p.p. type,
ii) $q_\lambda(\langle \vec{0}, d\rangle) \in J - J'$.

Let $\psi'(u_0, \cdots, u_{h'-1}, y)$ be the formula

$$\varphi(\vec{0}, y) \,\&\, \bigwedge_{i<h'} \neg\psi_\lambda(\vec{0}, y - u_i).$$

Clearly $M \models \psi'(d_0, \cdots, d_{h'-1}, d)$ if and only if $d \in N$ and $q_\lambda(\langle \vec{0}, d\rangle) \notin J'$. Since the $d_i$'s lie in $A$, the $(n + h')$-tuples $\langle a_0, \cdots, a_{n-1}, d_0, \cdots, d_{h'-1}\rangle$, $\langle a'_0, \cdots, a'_{n-1}, d'_0, \cdots, d'_{h'-1}\rangle$ realize the same p.p. type. Finally note that $\psi'(\vec{u}, y)$ contains only $h' < h \leqq k$ conjuncts being negations of p.p. formulas. Combining the last two facts with the induction hypothesis and Lemma 4 (with $\psi'(\vec{u}, y)$ as $\psi(\vec{x}, y)$) we obtain elements $d, e \in N$ satisfying (i) and (ii). This concludes the proof of Lemma 1.

## 3. Abelian groups

When dealing with abelian groups we can replace p.p. formulas by formulas expressing divisibility. Call a formula $\varphi(x_0, \cdots, x_{n-1})$ a $d$-formula if it is of the form $\Sigma_{\nu<n} k_\nu x_\nu = 0$ or of the form $\exists y(p^k y = \Sigma_{\nu<n} k_\nu x_\nu)$ for some prime $p$, natural number $k$ and integers $k_\nu$. Combining the elementary Lemma 4.3 from [2] with the fact that an embedding between abelian groups preserves p.p. formulas if and only if it preserves $d$-formulas it is easy to see that two $n$-tuples $\vec{a}$, $\vec{a}'$ from some abelian group $M$ realize the same p.p. type if and only if they realize the same $d$-type. Applying Lemma 2 and compactness as in the proof of (*) we obtain the following weak form of Szmielew's elimination-of-quantifier-result:

Every formula in the language of abelian groups is equivalent relative to the theory of abelian groups to a boolean combination of $\forall\exists$-sentences and divisibility formulas.

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